Hidden Symmetries in Discrete Clifford Analysis

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- **Planck’s constant.**
- **kinetic energy operator**
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\[ V(x) = \frac{1}{2} \sum_{j=1}^{n} (\partial_{x_j} u(x))^2. \]

**Ladder operators:**
\[ a_j = \frac{1}{\sqrt{2}} \left( \partial_{x_j} u(x) + \frac{i}{\sqrt{m}} \partial_{x_j} \right) \quad \text{and} \quad a_j^\dagger = \frac{1}{\sqrt{2}} \left( \partial_{x_j} u(x) - \frac{i}{\sqrt{m}} \partial_{x_j} \right) \]
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Quantum Field Theory (QFT) setting

Exact solutions of the time-independent Schrödinger equation are determined by means of the 2nd quantization approach.

- Fock space: Vector space $(\mathcal{F}, \langle \cdot | \cdot \rangle)$ such that
  - $\mathcal{F}$: Free algebra generated by the elements $a_j$ and $a_j^\dagger$ from the vacuum vector $\phi$ such that $a_j \phi = 0$.
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- Standard lemma in QFT: All the basic vectors in $\mathcal{F}$ have the following form
  $$\phi_u(x) = \left( \prod_{j=1}^n (a_j^\dagger)_{\alpha_j} \right) \phi(x)$$

- Vacuum vector connection: If $\phi(x)$ is a vacuum vector then
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and so, the time-harmonic Schrödinger equation is exactly solvable.
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Clifford Analysis: Study of operators belonging to the algebra

\[ \text{Alg} \{ x_j, \partial x_j, e_j : j = 1, \ldots, n \}, \]

1. \( x_j \) and \( \partial x_j \) satisfy the Weyl-Heisenberg graded commuting relations

\[ [\partial x_j, \partial x_k] = [x_j, x_k] = 0 \text{ and } [\partial x_j, x_k] = \delta_{jk} I. \]

2. \( e_1, e_2, \ldots, e_n \) are the generators of the Clifford algebra \( C\ell_{0,n} \). The remainder graded anti-commuting relations are given by

\[ e_j e_k + e_k e_j = -2\delta_{jk}. \]

- Multivector derivative: \( D = \sum_{j=1}^n e_j \partial x_j \) is the standard Dirac operator (embedding of the gradient derivative on \( C\ell_{0,n} \)).

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Basic operators and relations

- **Basic properties:**
  1. Laplacian splitting: \( \Delta := \sum_{j=1}^{n} \partial_{x_j}^2 = -D^2 \)
  2. Exact solvability: Under the existence of a function \( u(x) \) satisfying \( V(x) = -\frac{1}{2} [Du(x)]^2 \) the time-harmonic Schrödinger equation is solvable.
  3. Ladder operator splitting: \( \mathcal{H} = -\frac{1}{2} (AA^\dagger + A^\dagger A) \), with

\[
A = \frac{1}{\sqrt{2}} \left( Du(x)I + \frac{\hbar}{\sqrt{m}} D \right) \quad \text{and} \\
A^\dagger = \frac{1}{\sqrt{2}} \left( Du(x)I - \frac{\hbar}{\sqrt{m}} D \right).
\]

- Wigner quantal symmetries: \( \text{span} \left\{ \frac{1}{2} A^2, \frac{1}{2} (A^\dagger)^2, \mathcal{H} \right\} \oplus \text{span} \left\{ A, A^\dagger \right\} \) is isomorphic to a Lie superalgebra \( \mathfrak{osp}(1|2) \) (cf. N.F. & G. Ren (2011)).
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  2. **Exact solvability:** Under the existence of a function \( u(x) \) satisfying \( V(x) = -\frac{1}{2} [Du(x)]^2 \) the time-harmonic Schrödinger equation is solvable.
  3. **Ladder operator splitting:** \( H = -\frac{1}{2} (AA^\dagger + A^\dagger A) \), with

\[
A = \frac{1}{\sqrt{2}} \left( Du(x)I + \frac{\hbar}{\sqrt{m}} D \right)
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- **Wigner quantal symmetries:**
  \( \text{span} \left\{ \frac{1}{2} A^2, \frac{1}{2} (A^\dagger)^2, H \right\} \oplus \text{span} \left\{ A, A^\dagger \right\} \) is isomorphic to a Lie superalgebra \( \mathfrak{osp}(1|2) \) (cf. N.F. & G. Ren (2011)).
Basic operators and relations

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Lie-algebraic discretization approaches enclosed on this talk:

- **Radial-type approach:** Starting from the Weyl-Heisenberg symmetries, this approach provides a way to represent the algebra of Clifford vector-valued polynomials as a set of finite difference operators possessing $\mathfrak{osp}(1|2)$ symmetries.

- **SUSY QM type approach:** A pair of finite difference operators $(A_h, A_h^\dagger)$, obtained from the knowledge of a finite difference discretization $u_h(x)$ of $u(x)$, provides a way to discretize the Hamiltonian operator $\mathcal{H}$ as $\mathcal{H}_h = -\frac{1}{2} \left( A_h A_h^\dagger + A_h^\dagger A_h \right)$. 
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Finite difference toolbox

1. **Equidistant lattice with mesh width** $h > 0$:

$$h\mathbb{Z}^n = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \frac{x}{h} \in \mathbb{Z}^n \right\}$$

2. **Forward/backward finite difference operators**

$$\left( \partial_h^{+j} f \right)(x) = \frac{f(x + h e_j) - f(x)}{h}, \quad \left( \partial_h^{-j} f \right)(x) = \frac{f(x) - f(x - h e_j)}{h}.$$ 

3. **Translation property**: $\partial_h^{+j}$ and $\partial_h^{-j}$ are interrelated by

$$(T_h^{\pm j} f)(x) = f(x \pm h e_j)$$ i.e.

$$T_h^{-j}(\partial_h^{+j} f)(x) = (\partial_h^{-j} f)(x) \quad \text{and} \quad T_h^{+j}(\partial_h^{-j} f)(x) = (\partial_h^{+j} f)(x).$$

4. **Product rules for finite difference operators**:

$$\partial_h^{+j} (g(x)f(x)) = (\partial_h^{+j} g)(x)f(x + h e_j) + g(x)(\partial_h^{+j} f)(x)$$

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Radial-type discretization
Lie-algebraic formulation

Radial-type approach: Study of finite difference operators belonging to the algebra

$$\text{Alg} \{ L_j, M_j, e_j : j = 1, \ldots, n \},$$

1. $L_j$ and $M_j$ are position and momentum operators, respectively, satisfying the set of Weyl-Heisenberg algebra relations

$$[L_j, L_k] = [M_j, M_k] = 0 \quad \text{and} \quad [L_j, M_k] = \delta_{jk} I$$

2. $e_1, e_2, \ldots, e_n$ are the generators of the Clifford algebra of signature $(0, n)$.

Multivector operators: Basic left endomorphisms acting that act on functions with values on $\mathcal{C}_{\ell_0,n}$.

- Multivector derivative: $L = \sum_{j=1}^{n} e_j L_j$ stands the Lie-algebraic counterpart of the Dirac operator $D = \sum_{j=1}^{n} e_j \partial x_j$.

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Radial-type discretization

Examples

1. **Forward finite differences:** The set of operators $\partial^{+j}_h$ and $x_j T^{-j}_h : f(x) \mapsto x_j f(x - h e_j)$ span the Weyl-Heisenberg algebra of dimension $2n + 1$. Moreover $D^+_h = \sum_{j=1}^{n} e_j \partial^{+j}_h$ and $X_h = \sum_{j=1}^{n} e_j x_j T^{-j}_h$ are the corresponding multivector ladder operators on the lattice $h \mathbb{Z}^n$.

2. **Backward finite differences:** $\partial^{-j}_h$ and $x_j T^{+j}_h : f(x) \mapsto x_j f(x + h e_j)$ also span the Weyl-Heisenberg algebra of dimension $2n + 1$. This turns out $D^-_h = \sum_{j=1}^{n} e_j \partial^{-j}_h$ and $X^-_h = \sum_{j=1}^{n} e_j x_j T^{+j}_h$ as the corresponding multivector ladder operators on the lattice $h \mathbb{Z}^n$.

3. **Discretization of the Hermite operator:** $D^+_h$ and $X_h - D^-_h$ is obtained from the set of ladder operators $L_j = \partial^{+j}_h$ and $L_j = x_j T^{-j}_h - \partial^{-j}_h$. Moreover $X_h - D^-_h$ approximates the Hermite operator $X - D = -\exp\left(\frac{|x|^2}{2}\right) D \exp\left(-\frac{|x|^2}{2}\right)$, as $h$ tends to zero (cf. N.F., arXiv:1402.2268).
Radial-type discretization
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1. **Forward finite differences**: The set of operators $\partial_h^{+j}$ and $x_j T_h^{-j} : f(x) \mapsto x_j f(x - h e_j)$ span the Weyl-Heisenberg algebra of dimension $2n + 1$. Moreover $D_h^+ = \sum_{j=1}^n e_j \partial_h^{+j}$ and $X_h = \sum_{j=1}^n e_j x_j T_h^{-j}$ are the corresponding multivector ladder operators on the lattice $h \mathbb{Z}^n$.

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Radial-type discretization
The Exponential Generating Function (EGF) approach

Many degrees of freedom for choose discretization operators:
(cf. N.F., SIGMA 9 (2013), 065) The set of operators
\[(x_j + \frac{h}{2}) T^+_h : f(x) \mapsto (x_j + \frac{h}{2}) f(x + he_j)\] and
\[(x_j - \frac{h}{2}) T^-_h : f(x) \mapsto (x_j - \frac{h}{2}) f(x - he_j)\] satisfy
\[
\left[\partial^{-j}_h, \left( x_k + \frac{h}{2} \right) T^+_h \right] = \left[\partial^{+j}_h, \left( x_k - \frac{h}{2} \right) T^-_h \right] = \delta_{jk} I
\]

cf. N.F., 2014, arxiv.org

The EGF of the form
\[
G_h(x, y; \kappa) = \prod_{j=1}^n \frac{1}{\kappa \left( \frac{1}{h} \log \left( 1 + hy_j \right) \right)} \left( 1 + hy_j \right)^{x_j/h}
\]
yield the set of operators \( L_j = \partial^{+j}_h \) and \( M_j = \left( x_j - \kappa' (\partial x_j) \kappa (\partial x_j)^{-1} \right) T^-_h \) as generators of the Weyl-Heisenberg algebra of dimension \( 2n + 1 \). Moreover, they are unique.
Proposition (N.F., arXiv:1402.2268– Proposition 3.1)

Let $\kappa(t)$ defined as above and $X_h$ the multiplication operator. If there is a multi-variable function $\lambda(y)$ ($y \in \mathbb{R}^n$) such that

$$
\lambda \left( \frac{D_h^+ \exp(x \cdot y)}{\exp(x \cdot y)} \right) = \prod_{j=1}^{n} \kappa(y_j)
$$

then the Fourier dual $\Lambda_h$ of $D_h^+$ is given by

$$
\Lambda_h = X_h - \left[ \log \lambda(D_h^+) , x \right].
$$

Remark: The multi-variable function $\lambda(y)$ ($y \in \mathbb{R}^n$) always exists and it is given by

$$
\lambda(y) = \prod_{j=1}^{n} \kappa \left( \frac{1}{h} \log(1 + hy_j) \right).
$$
Why one needs $\mathfrak{su}(1, 1)$ based symmetries?

The Weyl-Heisenberg symmetry breaking

Main Goal:

For a given polynomial $w(t)$ of degree 1, with $\mu = \partial_h^{+j} w(x_j) = \partial_h^{-j} w(x_j)$, study the spectra of the coupled eigenvalue problem

$$E_h^+ f(x) = E_h^- f(x) = \varepsilon f(x)$$

carrying $E_h^\pm = \sum_{j=1}^n \mu^{-1} w(x_j \pm \frac{h}{2}) \partial_h^\pm$.

- **Drawback:** The set of operators $\partial_h^{+j}, \partial_h^{-j}, W_h^{-j} = \mu^{-1} w(x_j + \frac{h}{2}) T_h^{-j}, W_h^{+j} = \mu^{-1} w(x_j + \frac{h}{2}) T_h^{+j}$ and $I$, with $j = 1, 2, \ldots, n$, do not endow a canonical realization of an Weyl-Heisenberg type algebra of dimension $4n + 1$.

- **Fill the Weyl-Heisenberg gap:** The set of operators $W_h^{-j} = \mu^{-1} w(x_j + \frac{h}{2}) T_h^{-j}, W_h^{+j} = \mu^{-1} w(x_j + \frac{h}{2}) T_h^{+j}$ and $W_j = \mu^{-1} w(x_j) I$ generate a Lie algebra isomorphic to $\mathfrak{sl}(2n, \mathbb{R})$ (N.F., SIGMA 9 (2013), 065–Lemma 1).
Hidden Symmetries in Discrete Clifford Analysis

Nelson Faustino

Starting with the Harmonic Oscillator

Exact solvability

Quantum Field setting

The multivector calculus approach

Lie-algebraic discretization

The Setting

Weyl-Heisenberg symmetries

\( su(1, 1) \) symmetries

The factorization approach

Overview

The star-Laplacian

The SUSY QM picture

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$su(1, 1) \cong sl(2, \mathbb{R})$. The remaining commuting relations are given by

\[
\begin{bmatrix}
\frac{W_h^+}{h}, \frac{W_h^-}{h}
\end{bmatrix} = \frac{W_h^+}{h}, \quad \begin{bmatrix}
\frac{W_h^-}{h}, \frac{W_h^+}{h}
\end{bmatrix} = -\frac{W_h^-}{h}, \quad \begin{bmatrix}
\frac{W_h^+}{h}, \frac{W_h^-}{h}
\end{bmatrix} = \frac{2}{h} W.
\]

Casimir operator: The operator of the form

\[
K_h = \left( \frac{W}{h} \right)^2 - \frac{1}{2} \left( \frac{W_h^+}{h} \frac{W_h^-}{h} + \frac{W_h^-}{h} \frac{W_h^+}{h} \right)
\]

determines an irreducible unitary representation $\pi_\lambda$ of $SU(1, 1)$ on the enveloping algebra $U(su(1, 1))$. This representation is labeled by the eigenvalues $\lambda$ of $K_h$. 

Discrete series representations of $SU(1, 1)$

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Ladder operators on \( h\mathbb{Z}^n \): \( W_h^+ = \sum_{j=1}^n W_{hj}^+ \), \( W_h^- = \sum_{j=1}^n W_{hj}^- \) and \( W = \sum_{j=1}^n W_j \) generate a Lie algebra isomorphic to \( su(1, 1) \cong sl(2, \mathbb{R}) \). The remaining commuting relations are given by

\[
\begin{align*}
\left[ \frac{W_h^+}{h}, \frac{W}{h} \right] &= \frac{W_h^+}{h}, \\
\left[ \frac{W_h^-}{h}, \frac{W}{h} \right] &= -\frac{W_h^-}{h}, \\
\left[ \frac{W_h^+}{h}, \frac{W_h^-}{h} \right] &= \frac{2}{h} W.
\end{align*}
\]

Casimir operator: The operator of the form

\[
K_h = \left( \frac{W}{h} \right)^2 - \frac{1}{2} \left( \frac{W_h^+}{h} \frac{W_h^-}{h} + \frac{W_h^-}{h} \frac{W_h^+}{h} \right)
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determines an irreducible unitary representation \( \pi_\lambda \) of \( SU(1, 1) \) on the enveloping algebra \( U(su(1, 1)) \). This representation is labeled by the eigenvalues \( \lambda \) of \( K_h \).
Discrete series representations of $SU(1, 1)$
Positive/Negative series representations

1. **Positive series representation of $SU(1,1)$:** $\pi_\lambda^+$ is thus determined by the set of ladder operators

\[
\begin{align*}
\pi_\lambda^+ \left( \frac{W_h^-}{h} \right) &= E_h^+ - E_h^- \\
\pi_\lambda^+ \left( \frac{W_h^+}{h} \right) &= \frac{W_h^+}{h} \\
\pi_\lambda^+ (W) &= E_h^+ + \frac{n}{2} I \\
\pi_\lambda^+ (K_h) &= \left( E_h^+ + \frac{n}{2} I \right) \left( E_h^+ + \left( \frac{n}{2} - 1 \right) I \right) - \frac{W_h^+}{h} (E_h^+ - E_h^-)
\end{align*}
\]

2. **Negative series representation of $SU(1,1)$:** $\pi_\lambda^-$ is thus determined by the set of ladder operators

\[
\begin{align*}
\pi_\lambda^- \left( \frac{W_h^-}{h} \right) &= \frac{W_h^-}{h} \\
\pi_\lambda^- \left( \frac{W_h^+}{h} \right) &= E_h^+ - E_h^- \\
\pi_\lambda^- (W) &= -E_h^- - \frac{n}{2} I \\
\pi_\lambda^- (K_h) &= \left( E_h^- + \frac{n}{2} I \right) \left( E_h^- + \left( \frac{n}{2} - 1 \right) I \right) - \frac{W_h^-}{h} (E_h^+ - E_h^-)
\end{align*}
\]
Discrete series representations of $SU(1, 1)$

Positive/Negative series representations

1. Positive series representation of $SU(1,1)$: $\pi^+_\lambda$ is thus determined by the set of ladder operators

\[
\begin{align*}
\pi^+_\lambda \left( \frac{W^-_h}{h} \right) &= E^+_h - E^-_h \\
\pi^+_\lambda \left( \frac{W^+_h}{h} \right) &= \frac{W^+_h}{h} \\
\pi^+_\lambda \left( \frac{W}{h} \right) &= E^+_h + \frac{n}{2} I \\
\pi^+_\lambda (K_h) &= (E^+_h + \frac{n}{2} I) (E^+_h + (\frac{n}{2} - 1) I) - \frac{W^+_h}{h} (E^+_h - E^-_h)
\end{align*}
\]

2. Negative series representation of $SU(1,1)$: $\pi^-_\lambda$ is thus determined by the set of ladder operators

\[
\begin{align*}
\pi^-_\lambda \left( \frac{W^-_h}{h} \right) &= \frac{W^-_h}{h} \\
\pi^-_\lambda \left( \frac{W^+_h}{h} \right) &= E^+_h - E^-_h \\
\pi^-_\lambda \left( \frac{W}{h} \right) &= -E^-_h - \frac{n}{2} I \\
\pi^-_\lambda (K_h) &= (E^-_h + \frac{n}{2} I) (E^-_h + (\frac{n}{2} - 1) I) - \frac{W^-_h}{h} (E^+_h - E^-_h)
\end{align*}
\]
Invariant subspaces $\mathcal{H}_{s;h}$: Spaces with basic polynomials of the form $w_s(x; h) = \left( \frac{W_+}{h} \right)^s m_0(x; h)$, with $E^+_h m_0(x; h) = E^-_h m_0(x; h) = 0$.

Invariant subspaces $\mathcal{H}_{s;-h}$: Spaces with basic polynomials of the form $w_s(x; -h) = \left( \frac{W_-}{h} \right)^s m_0(x; h)$, with $E^+_h m_0(x; h) = E^-_h m_0(x; h) = 0$.

Irreducible subspaces: The $SO(n)$ invariant subspaces of the form $\left( \frac{W_+}{h} \right)^r (\mathcal{H}_{s-r;h} \cap \mathcal{H}_{s-r;-h})$ resp. $\left( \frac{W_-}{h} \right)^r (\mathcal{H}_{s-r;h} \cap \mathcal{H}_{s-r;-h})$.

Fourier decomposition of $\mathcal{H}_{s;\pm h}$: is determined from the Howe dual pair $(SO(1, 1), su(1, 1))$. as a direct sum of the $(s + 1)$ irreducible pieces $\left( \frac{W_{\pm}}{h} \right)^r (\mathcal{H}_{s-r;h} \cap \mathcal{H}_{s-r;-h})$. 
● **Invariant subspaces** $\mathcal{H}_{s; h}$: Spaces with basic polynomials of the form $w_s(x; h) = \left(\frac{w^+_h}{h}\right)^s m_0(x; h)$, with

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**The Howe dual pair technique**

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Overview

Why must we use Lie-algebraic based discretizations?

**Lie-algebraic based discretizations:**

1. **Preserve canonical symmetries:** Get exact representation formulae for the polynomial solutions from methods already known in *continuum*;

2. **Deep understanding of physical models:** Provides a general scheme to construct sequences of polynomials as eigenfunctions of a discrete Hamiltonian operator.

3. **Application to Cauchy problems:** The 1-parameter representation of $SU(1, 1)$ produces solutions of homogeneous Cauchy-problems as hypergeometric series representations (cf. *N.F.*, *SIGMA* 065, 2013–Section 4).

4. **Provides an operational framework:** The construction polynomials on the lattice based on the knowledge of the EGF makes intuitive and fully rigorous the study special functions and integral transforms on the lattice (cf. *N.F.*, arXiv:1402.2268–Subsection 3.3).
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**Star Laplacian:**

\[
\Delta_h f(x) = \sum_{j=1}^{n} \frac{f(x + he_j) + f(x - he_j) - 2f(x)}{h^2}
\]

- **Finite difference representation:** \( \Delta_h = \sum_{j=1}^{n} \frac{\partial_{h}^{+j} \partial_{h}^{-j}}{h^2} = \sum_{j=1}^{n} \frac{1}{h^2} \left( \partial_{h}^{+j} - \partial_{h}^{-j} \right) \).
- **Using forward and backward Dirac operators:** \( \Delta_h = -\frac{1}{2} \left( D_{h}^{+} D_{h}^{-} + D_{h}^{-} D_{h}^{+} \right) \).
- **Using a central difference Dirac operator:** \( \Delta_h = -\frac{1}{4} \left( D_{h/2}^{+} + D_{h/2}^{-} \right)^2 \).
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**Figure:** The star laplacian in \( h\mathbb{Z}^3 \)
Star Laplacian:

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Star-Laplacian
Factorization in $\mathbb{C}_{0,n}$

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*Figure:* The star laplacian in $\hbar \mathbb{Z}^3$
Hamiltonian operator:

$$\mathcal{H}_h = -\frac{\hbar^2}{m} \Delta_h + V(M), \quad M = \sum_{j=1}^n e_j M_j$$

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  $$A_h = \frac{1}{\sqrt{2}} \left( D u(M) + \frac{\hbar}{2\sqrt{m}} \left( D^+_{h/2} + D^-_{h/2} \right) \right)$$

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Remarks:

- The Weyl-Heisenberg symmetries leave invariant the exact solvability of the system.
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Exact solvability by means of the Weyl-Heisenberg picture

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  A_{h}^\dagger = \frac{1}{\sqrt{2}} \left( D_u(M) - \frac{\hbar}{2\sqrt{m}} \left( D_{h/2}^+ + D_{h/2}^- \right) \right)
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- **Weyl-Heisenberg constraint:**

\[
\left[ \frac{1}{2} \left( \partial_{h/2}^+ + \partial_{h/2}^- \right), M_k \right] = \delta_{jk} I
\]

- **Ladder operators:**

\[
A_h = \frac{1}{\sqrt{2}} \left( D\mu(M) + \frac{\hbar}{2\sqrt{m}} \left( D_{h/2}^+ + D_{h/2}^- \right) \right)
\]

\[
A_h^\dagger = \frac{1}{\sqrt{2}} \left( D\mu(M) - \frac{\hbar}{2\sqrt{m}} \left( D_{h/2}^+ + D_{h/2}^- \right) \right)
\]

Remarks:

- The Weyl-Heisenberg symmetries leave invariant the exact solvability of the system.

- **Isotropy** is not preserved under the Weyl-Heisenberg discretization scheme.
Discrete Quantum Mechanics
The SUSY approach

Hamiltonian operator:

\[ \mathcal{H}_h = -\frac{\hbar^2}{m} J_h + V_h(x) \]

- Hamiltonian Constraint: \[ \mathcal{H}_h = -\frac{1}{2} \left( A_h A_h^\dagger + A_h^\dagger A_h \right) \]

- Ladder operators:

\[ A_h = \frac{1}{\sqrt{2}} \left( \left[ D_h^+, u_h(x) \right] + \frac{\hbar}{\sqrt{m}} D_h^+ \right) \]

\[ A_h^\dagger = \frac{1}{\sqrt{2}} \left( \left[ D_h^-, u_h(x) \right] - \frac{\hbar}{\sqrt{m}} D_h^- \right) \]

- Exact solvability constraint:

\[ V_h(x) = -\frac{1}{4} \left[ (D_h^+ u_h) (x) \right]^2 - \frac{1}{4} \left[ (D_h^- u_h) (x) \right]^2 \]

- Jacobi-type operator:

\[ J_h = \Delta_h + \frac{\sqrt{m}}{\hbar} \sum_{j=1}^{n} \left( \partial_h^{+j} u_h \right) (x) \partial_h^{+j} - \frac{\sqrt{m}}{\hbar} \sum_{j=1}^{n} \left( \partial_h^{-j} u_h \right) (x) \partial_h^{-j} \]
**Hamiltonian operator:**

\[
\mathcal{H}_h = -\frac{\hbar^2}{m} J_h + V_h(x)
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- **Hamiltonian Constraint:** \( \mathcal{H}_h = -\frac{1}{2} \left( A_h A_h^\dagger + A_h^\dagger A_h \right) \).

- **Ladder operators:**

\[
A_h = \frac{1}{\sqrt{2}} \left( [D_h^+, u_h(x)] + \frac{\hbar}{\sqrt{m}} D_h^+ \right)
\]

\[
A_h^\dagger = \frac{1}{\sqrt{2}} \left( [D_h^-, u_h(x)] - \frac{\hbar}{\sqrt{m}} D_h^- \right)
\]

- **Exact solvability constraint:**

\[
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\]

- **Jacobi-type operator:**

\[
J_h = \Delta_h + \frac{\sqrt{m}}{\hbar} \sum_{j=1}^{n} \left( \partial_h^{+j} u_h \right)(x) \partial_h^{+j} - \frac{\sqrt{m}}{\hbar} \sum_{j=1}^{n} \left( \partial_h^{-j} u_h \right)(x) \partial_h^{-j}
\]
Hamiltonian operator:

\[ \mathcal{H}_h = -\frac{\hbar^2}{m} J_h + V_h(x) \]

- **Hamiltonian Constraint:** \( \mathcal{H}_h = -\frac{1}{2} \left( A_h A^\dagger_h + A^\dagger_h A_h \right) \).
- **Ladder operators:**
  \[
  A_h = \frac{1}{\sqrt{2}} \left( [D^+_h, u_h(x)] + \frac{\hbar}{\sqrt{m}} D^+_h \right) \\
  A^\dagger_h = \frac{1}{\sqrt{2}} \left( [D^-_h, u_h(x)] - \frac{\hbar}{\sqrt{m}} D^-_h \right)
  \]
- **Exact solvability constraint:**
  \[
  V_h(x) = -\frac{1}{4} \left( [D^+_h u_h(x)] \right)^2 - \frac{1}{4} \left( [D^-_h u_h(x)] \right)^2
  \]
- **Jacobi-type operator:**
  \[
  J_h = \Delta_h + \frac{\sqrt{m}}{\hbar} \sum_{j=1}^n \left( \partial_h^+ j u_h \right) (x) \partial_h^+ - \frac{\sqrt{m}}{\hbar} \sum_{j=1}^n \left( \partial_h^- j u_h \right) (x) \partial_h^-
  \]
Discrete Quantum Mechanics
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Hamiltonian operator:

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- **Ladder operators:**
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Ongoing Research

- **Break of the Weyl-Heisenberg symmetries**: When \( u_h(x) \) is a polynomial of degree 2, the components of \( A_h \) and \( A_h^\dagger \),

\[
A_h^i = \frac{1}{\sqrt{2}} \left( \left( \partial_h^+ u_h \right)(x) T_h^+ + \frac{\hbar}{\sqrt{m}} \partial_h^+ \right) \quad \text{and} \quad A_h^{\dagger j} = \frac{1}{\sqrt{2}} \left( \left( \partial_h^- u_h \right)(x) T_h^- - \frac{\hbar}{\sqrt{m}} \partial_h^- \right),
\]

respectively, do not encode Weyl-Heisenberg symmetries – as in continuum – but instead \( su(1,1) \) symmetries.

- **Beyond the De Donder-Weyl approach**: \( z = q + ip \mapsto A_h \) and \( z^\dagger = q - ip \mapsto A_h^\dagger \), where \( q = \sum_{j=1}^{n} q_j e_j \) and \( p = \sum_{j=1}^{n} p_j e_j \) are 'polymomenta' representations for the phase space, produces a meson-type quantization (cf. G. Wentzel, *Quantum Theory of Fields*, (1949))

- **Open problems**: What kind of Lie-algebraic representations arise from 'De Donder-Weyl approach'? Are all they unitarily equivalent to \( su(1,1) \) or even to the Lie superalgebra \( sl(2|1) \) (see, for instance, Jafarov– Van der Jeugt recent papers/preprints)?
Ongoing Research

- **Break of the Weyl-Heisenberg symmetries:** When $u_h(x)$ is a polynomial of degree 2, the components of $A_h$ and $A_h^\dagger$,
  
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  $A_h^{\dagger j} = \frac{1}{\sqrt{2}} \left( (\partial^-_h u_h)(x) \right. T^-_h - \frac{\hbar}{\sqrt{m}} \partial^-_h \left. \right)$, respectively, do not encode Weyl-Heisenberg symmetries – as in continuum – but instead $su(1, 1)$ symmetries.

- **Beyond the De Donder-Weyl approach:** $z = q + ip \mapsto A_h$ and $z^\dagger = q - ip \mapsto A_h^\dagger$, where $q = \sum_{j=1}^n q_je_j$ and $p = \sum_{j=1}^n p_je_j$ are 'polymomenta' representations for the phase space, produces a meson-type quantization (cf. G. Wentzel, *Quantum Theory of Fields*, (1949))

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**Ongoing Research**

- **Break of the Weyl-Heisenberg symmetries:** When \( u_h(x) \) is a polynomial of degree 2, the components of \( A_h \) and \( A_h^\dagger \),

\[
A_h^i = \frac{1}{\sqrt{2}} \left( (\partial^+ h u_h) (x) \ T^+_h + \frac{\hbar}{\sqrt{m}} \partial^+_h \right)
\]

and

\[
A_h^{\dagger j} = \frac{1}{\sqrt{2}} \left( (\partial^- h u_h) (x) \ T^-_h - \frac{\hbar}{\sqrt{m}} \partial^-_h \right),
\]

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‘FAPESP is a very interesting model for us because São Paulo is one of the few states in the world where support of research is linked directly to gross domestic product (GDP),’ *Martyn Poliakoff*, vice-president of the Royal Society (UK), [Nature 510, 201 (12 June 2014) doi:10.1038/510201a]